STEADY FLOWS OF AN ANISOTROPICALLY CONDUCTING MEDIUM IN A HALF-SPACE

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S.A. REGIRER and I.B. CHEKMAREV
(Moscow, Leningrad)

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The simplest problem of motion of an anisotropic conducting medium in a half-space in the presence of a magnetic field was examined by Fay [1] and Sakhanovskii [2], who established that the distribution of velocities in the fluid is not monotone. A periodic structure of the velocity distribution, arising as a result of anisotropy of conductivity is apparently characteristic for many flow problems in semi-bounded regions. Some such problems are examined below.

1. Let us first examine the steady flow of a conducting medium in translational motion above an infinite plane. The external magnetic field B_0 is considered homogeneous and perpendicular to the plane. We assume that the plane is permeable and that in this plane the values of all components of velocity, the temperature and of the magnetic field are given. At the infinite distance from the plane all quantities are assumed to be bounded, and some of them (components of velocity) are assumed to be known, while the remaining ones are determined from the solution. We shall use Ohm's law in the form applicable to a weakly ionized gas when the anisotropy of conductivity is reduced to the Hall effect. The dependence of physical properties of the medium on temperature and pressure will be assumed known.

The x- and z-axes are oriented in the plane while the y-axis is perpendicular to it. We assume that all quantities depend on y only. From general equations of magnetohydrodynamics we then obtain the following system [3]:

$$\frac{d}{dy}(\rho v) = 0, \quad \rho v \frac{dv}{dy} = -\frac{dp}{dy} + \frac{4}{3}\frac{d}{dy}\left(\eta \frac{dv}{dy}\right) + \frac{1}{c}(j_z B_x - j_x B_z) \qquad (1.1)$$

$$\rho v \frac{du}{dy} = \frac{d}{dy}\left(\eta \frac{du}{dy}\right) - \frac{1}{c} j_z B_0, \qquad \rho v \frac{dw}{dy} = \frac{d}{dy}\left(\eta \frac{dw}{dy}\right) + \frac{1}{c} j_x B_0 \qquad (1.2)$$

$$j_x = \frac{c}{4\pi}\frac{dB_z}{dy}, \quad j_y = 0 \quad j_z = -\frac{c}{4\pi}\frac{dB_x}{dy}, \quad \frac{dE_x}{dy} = 0, \quad \frac{dE_z}{dy} = 0 \tag{1.3}$$

$$j_x = \sigma\left[E_x + \frac{1}{c}(vB_z - wB_0)\right] + \alpha j_z B_0$$

$$\sigma\left[E_y + \frac{1}{c}(wB_x - uB_z)\right] + \alpha(j_x B_z - j_z B_x) = 0 \tag{1.4}$$

$$j_z = \sigma\left[E_z + \frac{1}{c}(uB_0 - vB_x)\right] - \alpha j_x B_0$$

$$F(p, \rho, T) = 0, \eta = \eta(p, T), k = k(p, T), \sigma = \sigma(p, T), \alpha = \frac{\omega_e\tau_e}{|B|} = \alpha(p, T) \tag{1.5}$$

$$\rho v c_v \frac{dT}{dy} = \frac{d}{dy}\left(k\frac{dT}{dy}\right) + \frac{1}{\sigma}(j_x^2 + j_z^2) +$$

$$+ \eta\left[\left(\frac{du}{dy}\right)^2 + \left(\frac{dw}{dy}\right)^2 + \frac{4}{3}\left(\frac{dv}{dy}\right)^2\right] - T\frac{dv}{dy}\left(\frac{\partial p}{\partial T}\right)_\rho \quad (c_v = c_v(p, T)) \tag{1.6}$$

Here α is the Hall coefficient, while the remaining symbols are the ones generally accepted. We shall also denote values of function at $y = 0$ and at $y \to \infty$ by indices w and ∞, respectively. From Equations (1.1), (1.3) and (1.4) we obtain

$$\rho v = \rho_w v_w = \rho_\infty v_\infty \tag{1.7}$$

$$E_x = -\frac{1}{c}(v_\infty B_{z\infty} - w_\infty B_0), \qquad E_z = -\frac{1}{c}(u_\infty B_0 - v_\infty B_{x\infty}) \tag{1.8}$$

The energy equation is transformed in the usual manner by adding it to equations of motion (1.1) and (1.2) multiplied by u, v and w. Utilizing relationships (1.7) and (1.8) and expressions for j_x and j_z from (1.2) we find

$$\rho_\infty v_\infty\left[c_v(T - T_\infty) + \frac{1}{2}(\mathrm{v}^2 - \mathrm{v}_\infty^2)\right] = k\frac{dT}{dy} + \frac{\eta}{2}\frac{d\mathrm{v}^2}{dy} + \frac{2}{3}\eta\frac{dv^2}{dy} -$$

$$- \frac{1}{B_0}\left\{\left[\rho_\infty v_\infty(w - w_\infty) - \eta\frac{dw}{dy}\right](v_\infty B_{z\infty} - w_\infty B_0) + \right.$$

$$\left. + \left[\rho_\infty v_\infty(u - u_\infty) - \eta\frac{du}{dy}\right](v_\infty B_{x\infty} - u_\infty B_0)\right\} + \tag{1.9}$$

$$+ \int_y^\infty\left[v\frac{dp}{dy} + T\frac{dv}{dy}\left(\frac{\partial p}{\partial T}\right)_\rho\right]dy$$

Further we shall consider the special case of an incompressible fluid with the following constant properties*:

$$\eta = \text{const}, \ k = \text{const}, \ \sigma = \text{const}, \ \alpha = \text{const}, \ \rho = \text{const}, \text{ and } c_v = \text{const}.$$

Then the solution to the system (1.1) to (1.6) can be obtained in the finite form. Let us introduce the notations

* This problem was first examined by I.P. Semenova (II Vsesoiuznyi s'ezd po teor. i prikl. mekhanike (II All-Union congress of theoret, and appl. mechanics) M., Ann. dokl., p. 194, 1964.

$$V^\circ = u + iw, \quad J^\circ = j_x + ij_z, \quad E^\circ = E_x + iE_z, \quad B^\circ = B_x + iB_z \tag{1.10}$$

and rewrite the initial equations in the following form

$$\frac{dp}{dy} = -\frac{1}{8\pi}\frac{d}{dy}|B^\circ|^2, \qquad E_y = \frac{\alpha c}{8\pi\sigma}\frac{d}{dy}|B^\circ|^2 - \frac{1}{c}\,\mathrm{Im}\,(V^\circ\overline{B}^\circ) \tag{1.11}$$

$$\rho v\frac{dV^\circ}{dy} = \eta\frac{d^2V^\circ}{dy^2} + \frac{iB_0}{c}J^\circ, \qquad J^\circ = \frac{ci}{4\pi}\frac{dB^\circ}{dy}, \qquad \frac{dE^\circ}{dy} = 0 \tag{1.12}$$

$$J^\circ = \sigma\left(E^\circ + \frac{iB_0}{c}V^\circ - \frac{iv}{c}B^\circ\right) - i\alpha B_0 J^\circ \tag{1.13}$$

$$\rho v c_v\frac{dT}{dy} = k\frac{d^2T}{dy^2} + \frac{1}{\sigma}|J^\circ|^2 + \eta\left|\frac{dV^\circ}{dy}\right|^2 \tag{1.14}$$

According to (1.7) here $v = v_w$. Following [4], we obtain

$$\frac{d^2V^\circ}{dy_1^2} - (R + R_m{}^\circ)\frac{dV^\circ}{dy_1} + (V^\circ - V_\infty{}^\circ)(RR_m{}^\circ - M^{\circ 2}) = 0$$

$$R = \frac{v\rho L}{\eta}, \quad R_m{}^\circ = \frac{4\pi v\sigma L}{c^2(1 + i\alpha B_0)}, \quad M^{\circ 2} = \frac{\sigma B_0{}^2L^2}{c^2\eta(1 + i\alpha B_0)}, \quad y_1 = \frac{y}{L} \tag{1.15}$$

Here L is some given quantity. It is evident that the general solution of this equation is

$$V^\circ = V_\infty{}^\circ + C_1e^{\gamma_1 y_1} + C_2e^{\gamma_2 y_1} \tag{1.16}$$

$$\gamma_{1,2} = s_{1,2} + i\omega_{1,2} = {}^1/_2(R + R_m{}^\circ) \pm {}^1/_2\sqrt{(R + R_m{}^\circ)^2 - 4(RR_m{}^\circ - M^{\circ 2})}$$

Further it is easy to show that the magnetic field, the current and other quantities are determined with accuracy to up to the constant values, by the following equations

$$B^\circ = B_\infty{}^\circ + \frac{4\pi\eta}{B_0L}[C_1(R - \gamma_1)e^{\gamma_1 y_1} + C_2(R - \gamma_2)e^{\gamma_2 y_1}]$$

$$J^\circ = -\frac{ic\eta}{B_0L^2}[C_1\gamma_1(R - \gamma_1)e^{\gamma_1 y_1} + C_2\gamma_2(R - \gamma_2)e^{\gamma_2 y_1}] \tag{1.17}$$

and relationships (1.11) and (1.14). Constant C_1 and C_2 must be determined from the conditions at $y_1 = 0$ and $y_1 \to \infty$. In connection with this we shall examine the quantities γ_1, and γ_2. After few simple transformations we find

$$s_{1,2} = \frac{a}{2} \pm \left\{\frac{a^2 - d^2 - 4b}{8} + \left[\left(\frac{a^2 - d^2 - 4b}{8}\right)^2 + \left(\frac{ad - 2\alpha B_0 b}{4}\right)^2\right]^{1/2}\right\}^{1/2} \tag{1.18}$$

$$\omega_{1,2} = -\frac{d}{2} \pm \left\{-\frac{a^2 - d^2 - 4b}{8} + \left[\left(\frac{a^2 - d^2 - 4b}{8}\right)^2 + \left(\frac{ad - 2\alpha B_0 b}{4}\right)^2\right]^{1/2}\right\}^{1/2}$$

$$a = R + R_{me}, \quad b = RR_{me} - M_e{}^2, \quad d = \alpha B_0 R_{me}, \quad M_e = |M^\circ|, \quad R_{me} = |R_m{}^\circ|$$

It is easy to show that $s_{1,2}$ become equal to zero only when $b = 0$, while $\omega_{1,2}$ only when $\alpha B_0 = 0$. Signs of $s_{1,2}$ with reference to v and b are determined according to the table shown on the left.

v	<0		>0		$=0$
b	$\geqslant 0$	>0	$\geqslant 0$	<0	$=-M_e^2$
s_1	$\leqslant 0$	>0	>0		>0
s_2	<0		$\geqslant 0$	<0	<0

This result is independent of values of other parameters, including αB_0 and therefore coincides with the result found ealier for the isotropically conducting fluid [5]. Similarly, the conclusions regarding the selection of constants C_1 and C_2 and also their relationship to the quantities V^0 and B^0 at $y = 0$ and $y \to \infty$, remain in force.

Thus for $v < 0,\ b > 0$ we have

$$C_1 = \frac{B_0 (B_w{}^\circ - B_\infty{}^\circ) L - 4\pi\eta (V_w{}^\circ - V_\infty{}^\circ)(R - \gamma_1)}{4\pi\eta(\gamma_1 - \gamma_2)}$$

$$C_2 = \frac{4\pi\eta (V_w{}^\circ - V_\infty{}^\circ)(R - \gamma_2) - B_0 (B_w{}^\circ - B_\infty{}^\circ) L}{4\pi\eta(\gamma_1 - \gamma_2)} \qquad (1.19)$$

where $V_w{}^\circ, V_\infty{}^\circ, B_w{}^\circ$, and $B_\infty{}^\circ$ can be given arbitrarily. For $v < 0,\ b \leqslant 0$ or $v > 0,\ b < 0$ we obtain respectively

$$C_1 = 0, \qquad C_2 = V_w{}^\circ - V_\infty{}^\circ, \qquad V_w{}^\circ - V_\infty{}^\circ = \frac{B_0 L}{4\pi\eta}\frac{B_w{}^\circ - B_\infty{}^\circ}{R - \gamma_2}$$

$$C_1 = V_w{}^\circ - V_\infty{}^\circ, \qquad C_2 = 0, \qquad V_w{}^\circ - V_\infty{}^\circ = \frac{B_0 L}{4\pi\eta}\frac{B_w{}^\circ - B_\infty{}^\circ}{R - \gamma_1} \qquad (1.20)$$

i.e. only three quantities can be set arbitrarily and only such flows can be realized for which $V_w{}^\circ - V_\infty{}^\circ$ and $B_w{}^\circ - B_\infty{}^\circ$ satisfy the relationships given in (1.20).

Finally, for $v > 0,\ b \geqslant 0$ we obtain $C_1 = C_2 = 0$ with the result that only the trivial condition $V^0 = \text{const}$ and $B^0 = \text{const}$ is possible.

Thus distribution of $V^\circ(y)$, $B^\circ(y)$, and $J^\circ(y)$ in the general case, a non-monotone character. The frequency of periodic changes of these quantities is determined by Equation (1.18), while for $C_1 \neq 0,\ C_2 \neq 0$ the combination of two oscillating functions with different periods ω_1 and ω_2. At the same time as it is well known, the sum may either be a periodic function (with period not less than max (ω_1, ω_2)), or a non-periodic function. From Equations (1.18) it is evident that for any non-trivial solution (containing two or one exponential term) it is possible, by selection of the sign of the external field B_0, to obtain one of the frequencies such that

$$|\omega| \doteq \frac{|d|}{2} + \left\{ -\frac{a^2 - d^2 - 4b}{8} + \left[\left(\frac{a^2 - d^2 - 4b}{8} \right)^2 + \left(\frac{ad - 2\alpha B_0 b}{4} \right)^2 \right]^{1/2} \right\}^{1/2}$$

From this $|\omega| > |\,{}^1/_2\, d\,|$ follows. Therefore, if $\alpha \neq 0$ and the sign B_0 is chosen such that $d > 0$ for $v < 0$ and $b < 0$ and $d < 0$ for $v > 0$ and $b < 0$, then increase in $|R_m|$ results in arbitrarily large values of frequency. On the other hand, when $|\alpha B_0| \to 0$ and $|\alpha B_0| \to \infty$ we have $\omega_{1,2} \to 0$. Therefore, only limited and, apparently insignificant change of frequencies can result from increase of the Hall coefficient.

It is interesting to compare the 'wave lengths' on the velocity profile, i.e. quantities $\lambda = 2\pi / \omega$, with the thickness δ of the boundary layer on the wall. It turns out that such flows are possible when $\lambda \ll \delta$. In fact, for example, let $C_2 \equiv 0$ by virtue of boundary conditions ($B_w^\circ - B_\infty^\circ$ in this case is related to γ_1 and γ_2), $v < 0$, $b > 0$, $B_0 > 0$ (i.e. $a < 0$, and $d < 0$). We also assume for the sake of simplicity

$$V_w^\circ = 0, \qquad V_\infty^\circ = u_\infty \qquad (w_\infty = 0)$$

Then

$$u = \operatorname{Re} V^\circ = u_\infty (1 - e^{s_1 y_1} \cos \omega_1 y_1) \tag{1.21}$$

Extrema of $u(y)$ are reached at the points where $\tan \omega_1 y_1 = s_1 / \omega_1$. For values of b close to zero the decrement s_1 is quite small while the frequency $\omega_1 \approx {}^1/_2 (|a| + |d|)$ may be large due to $|a| \gg |d|$. At the same time $|s_1 / \omega_1| \ll 1$ and points of the extremum are determined by the approximate relationship

$$y_1^{(k)} \approx \frac{\pi k + s_1/\omega_1}{\omega_1} \qquad (k = 1, 2, \ldots)$$

The velocity at the point $y_1 = y_1^{(k)}$ is, according to (1.21),

$$u(y^{(k)}) \approx u_\infty \left[1 - (-1)^k \cos(s_1/\omega_1) \exp \frac{s_1}{\omega_1}\left(\pi k + \frac{s_1}{\omega_1}\right)\right]$$

Neglecting the square of the small quantity s_1 / ω_1, we obtain from this

$$u(y^{(k)}) \approx u_\infty \left[1 - (-1)^k \cos \frac{s_1}{\omega_1} e^{\pi k s_1/\omega_1}\right]$$

In this manner, when $s_1 / \omega_1 \ll 1 / \pi k$, the velocity in the k-th extremum is different from u_∞, and therefore $\delta > y^{(k)} \gg y^{(2)} > \lambda$. On the other hand, if the displacement thickness is introduced in the usual way

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_\infty}\right) dy = \frac{Ls_1}{s_1^2 + \omega_1^2} \approx \frac{Ls_1}{\omega_1^2}$$

then obviously $y_1^{(k)} > \delta^* / L$ for all k. Consequently, for $\omega_1 \gg s_1$ the displacement in a given case does not characterize the thickness of the boundary layer in the sense that for $y > \delta^*$ the values of velocity may differ strongly from u_∞.

Returning to the energy equation (1.14) it is not difficult to show that the temperature distribution is always monotone since

$$k \frac{dT}{dy} = e^{\rho v c_v y/k} \int_y^\infty e^{-\rho v c_v y/k} \left(\frac{1}{\sigma} |J^\circ|^2 + \eta \left|\frac{dV^\circ}{dy}\right|^2\right) dy > 0$$

With increase in the parameters s and ω the thermal fluxes increase substantially. For the particular case examined above, where u is determined by equation (1.21) and

$w = - u_\infty e^{s_1 y_1} \sin \omega_1 y_1$, we have from (1.17)

$$|J^\circ|^2 = \left(\frac{c\eta}{B_0 L^2}\right)^2 u_\infty^2 (s_1^2 + \omega_1^2) \left[(R - s_1)^2 + \omega_1^2\right] e^{2 s_1 y_1}$$

i.e. $| k\, dT / dy |$ increases as ω_1^4.

2. The effect of variable properties and compressibility of fluid on the character of the velocity profile can be evaluated approximately by constructing, for example, the solution for the system (1.1) to (1.6) for large values of y. We write

$$\mathbf{v} = \mathbf{v}_\infty + \mathbf{v}', \quad \mathbf{B} = \mathbf{B}_\infty + \mathbf{B}', \quad \mathbf{j} = \mathbf{j}', \quad \mathbf{E} = \mathbf{E}_\infty + \mathbf{E}'$$

$$p = p_\infty + p', \quad \rho = \rho_\infty + \rho', \quad T = T_\infty + T', \quad \sigma = \sigma_\infty + \sigma'$$

$$k = k_\infty + k', \quad \eta = \eta_\infty + \eta', \quad \alpha = \alpha_\infty + \alpha', \quad c_v = c_{v\infty} + c_v'$$

where all quantities with the index ∞ are constant. Linearized system (1.1) to (1.6) after omitting the primes takes the form

$$\rho_\infty v + \rho v_\infty = 0 \tag{2.1}$$

$$\rho_\infty v_\infty \frac{du}{dy} = \eta_\infty \frac{d^2 u}{dy^2} - \frac{1}{c} j_z B_0, \qquad \rho_\infty v_\infty \frac{dw}{dy} = \eta_\infty \frac{d^2 w}{dy^2} + \frac{1}{c} j_x B_0 \tag{2.2}$$

$$\rho_\infty v_\infty \frac{dv}{dy} = -\frac{dp}{dy} + \frac{4}{3}\eta_\infty \frac{d^2 v}{dy^2} + \frac{1}{c}(j_z B_{x\infty} - j_x B_{z\infty}) \tag{2.3}$$

$$j_x = \frac{c}{4\pi}\frac{dB_z}{dy}, \qquad j_z = -\frac{c}{4\pi}\frac{dB_x}{dy} \tag{2.4}$$

$$j_x = \frac{\sigma_\infty}{c}(v B_{z\infty} + v_\infty B_z - w B_0) + \alpha_\infty j_z B_0 \tag{2.5}$$

$$j_z = \frac{\sigma_\infty}{c}(u B_0 - v_\infty B_x - v B_{x\infty}) - \alpha_\infty j_x B_0$$

$$\sigma\left[E_y + \frac{1}{c}(w B_{x\infty} + w_\infty B_x - u B_{z\infty} - u_\infty B_z)\right] = -\alpha_\infty (j_x B_{z\infty} - j_z B_{x\infty}) \tag{2.6}$$

$$l_1 p + l_2 \rho + l_3 T = 0, \qquad \rho_\infty v_\infty c_{v\infty}\frac{dT}{dy} = k_\infty \frac{d^2 T}{dy^2} + T_\infty \frac{l_3}{l_1}\frac{dv}{dy} \tag{2.7}$$

Here in writing (2.1) equation (1.7) was utilized. Constants l_1, l_2, and l_3 are the values of derivatives of F with respect to p, ρ, and T when $y \to \infty$. We note that equations (1.5) for η, k, σ, α, and c_v remained unused because the perturbations of these quantities do not enter into the equations and in the adopted approximation do not exert any influence on the asymptotic behavior of the flow. It should be pointed out that the weak influence of variation of properties is found also when a different approach to the problem is used. For an incompressible fluid, for example, it turns out, that for $v \equiv 0$ small values of $(d \ln \sigma / d \ln T)_\infty$ of the order of ε correspond to changes in frequencies ω of the next order of magnitude as compared with ε *

* This fact was established by L.E. Pekurovskii and I.M. Rutkevich, students of MGU.

We shall seek solutions of the system (2.1) to (2.7) in the form

$$u = A_u e^{\gamma y_1}, \quad v = A_v e^{\gamma y_1}, \ . \ . \ . \ , T = A_T e^{\gamma y_1}$$

with the condition that Re $\gamma < 0$. For determination of γ we then obtain the equation

$$\left\{\Lambda_2 - (\gamma - P)\left[\Lambda_1\left(\frac{4\gamma}{3} - R\right) - 1\right]\right\}\left\{\frac{\alpha_\infty^2 B_0^2 \gamma^2}{M^4}(R-\gamma)^2 + \left[1 - \frac{(R-\gamma)(R_m - \gamma)}{M^2}\right]^2\right\} =$$
$$= \Lambda_1 \frac{B_\infty^2}{B_0^2}(\gamma - P)(R-\gamma)\left[1 - \frac{(R-\gamma)(R_m-\gamma)}{M^2}\right] \qquad (2.8)$$
$$R = \frac{v_\infty \rho_\infty L}{\eta_\infty}, \qquad R_m = \frac{4\pi\sigma_\infty v_\infty L}{c^2}, \qquad P = \frac{v_\infty \rho_\infty c_{v\infty} L}{k_\infty}$$
$$M^2 = \frac{B_0^2 L^2 \sigma_\infty}{c^2 \eta_\infty}, \qquad \Lambda_1 = \frac{l_1 \eta_\infty v_\infty}{l_2 \rho_\infty L}, \qquad \Lambda_2 = \frac{l_3^2 T_\infty v_\infty L}{l_1 l_2 \rho_\infty k_\infty}$$

In the case of incompressible fluid ($\Lambda_1 = \Lambda_2 = 0$) we find, from (2.8), four values of γ, which naturally coincide with values of $\gamma_{1, 2}$ and their complex conjugates $\bar{\gamma}_{1, 2}$, computed in section 1. The effect of compressibility leads to the appearance of two additional roots. Besides, as a result of compressibility the asymptotic behavior of the solution begins to depend on the flow and field parameters at infinity.

The distribution of velocities has a periodic structure, and the frequencies are the same as in the case of incompressible fluid when $B_\infty = 0$ and $\Lambda_1 = 0$ (the density depends on temperature only), and also for $M_\infty^2 \ll 1$ in the case of perfect gas, when

$$M_\infty^2 = \frac{v_\infty^2}{\varkappa(\varkappa - 1) c_{v\infty} T_\infty}, \quad l_1 = 1, \quad l_2 = -c_{v\infty}(\varkappa - 1) T_\infty, \quad l_3 = -c_{v\infty}(\varkappa - 1)\rho_\infty$$
$$(\varkappa = c_p / c_v)$$

For $\Lambda_1 \neq 0$ and the finite values of other parameters an increase in B_∞^2 / B_0^2 leads to the result, that all six roots of equation (2.8) become real. When $B_\infty^2 / B_0^2 \to \infty$, then four of these roots approach finite values

$$P, R, {}^1/_2 (R + R_m) \pm {}^1/_2 \sqrt{(R + R_m)^2 - 4(RR_m - M^2)}$$

while the other two increase without bounds as

$$\pm \frac{B_\infty}{B_0 M^3} \sqrt{{}^4/_3 (1 + \alpha_\infty^2 B_0^2)}$$

In this manner the distribution of velocities in a compressible flow far from the boundary may be periodic as well as monotone. The latter is always achieved here in case of appropriate increase of the longitudinal magnetic field at infinity.

We note that the initial system of equations (1.1) to (1.6) coincides with equations describing the structure of a shock wave. If we limit ourselves to the case of an inviscid and thermally non-conducting perfect gas, we obtain in the limit, from equation (2.8)

$$\left(\frac{\gamma}{N}\right)^2 (1 + \alpha_\infty^2 B_0^2) + \left[2\left(1 - \frac{R_m}{N}\right) + \frac{B_\infty^2 M_\infty^2}{B_0^2 (1 - M_\infty^2)}\right]\frac{\gamma}{N} +$$
$$+ \left(1 - \frac{R_m}{N}\right)\left[\left(1 - \frac{R_m}{N}\right) + \frac{B_\infty^2 M_\infty^2}{B_0^2 (1 - M_\infty^2)}\right] = 0 \qquad (N = M^2 / R)$$

The discriminant of this equation

$$D=\left[2\left(1-\frac{R_m}{N}\right)+\frac{B_\infty^2M_\infty^2}{B_0^2(1-M_\infty^2)}\right]^2-$$
$$-4(1+\alpha_\infty^2B_0^2)\left(1-\frac{R_m}{N}\right)\left[\left(1-\frac{R_m}{N}\right)+\frac{B_\infty^2M_\infty^2}{B_0^2(1-M_\infty^2)}\right]$$

is analogous to Expression (23) of [6], where the structure of a shock wave in an anisotropically conducting gas was analyzed. The condition $D<0$, which is necessary and sufficient for the existence of periodic distribution of parameters in the flow above the plane, is consequently identical with the condition for the periodicity of structure. With the increase in B_∞^2/B_0^2, the periodicity, as was indicated, disappears. This is related to the change in the type of singular points of the initial non-linear system of equations [6].

3. Properties of the medium may, generally speaking, depend not only on the temperature and pressure but also on other quantities. If, in particular, initial equations (1.1) to (1.4) and (1.6) and the equation of state (1.5) are considered as 'single-fluid approximation' in the dynamics of an ionized gas, then the scalar conductivity σ may be considered as depending on the concentration of electrons which is determined from additional considerations. It is apparent that in this case a change in the conductivity will not have any effect on the periodic structure of the velocity profile far from the boundary. However, in contrast to section 2, the construction of approximate solution for the entire flow is here sometimes successful.

Let us examine for example the motion of a weakly ionized gas at low pressure where it is possible to neglect the induced magnetic field and processes of spatial ionization and recombination. In this case concentrations of charged particles near the wall at which recombination takes place are governed by ambipolar diffusion. Taking into account only the collisions of neutral particles with electrons and ions and neglecting, as usual, inertial terms, it is possible to write the following equations of motion of charged particles [3]:

$$\nabla p_e+en_e\left(\mathbf{E}+\frac{1}{c}\,\mathbf{v}\times\mathbf{B}\right)+\frac{en_e}{c}\,\mathbf{v}_e\times\mathbf{B}+\frac{m_en_e}{\tau_e}\,\mathbf{v}_e=0 \qquad (3.1)$$
$$\nabla p_i-en_i\left(\mathbf{E}+\frac{1}{c}\,\mathbf{v}\times\mathbf{B}\right)-\frac{en_i}{c}\,\mathbf{v}_i\times\mathbf{B}+\frac{m_in_i}{\tau_i}\,\mathbf{v}_i=0$$
$$\operatorname{div}\left[n_e(\mathbf{v}+\mathbf{v}_e)\right]=0 \qquad (3.2)$$

here n_e and n_i are the concentrations of electrons and ions, $\mathbf{v}_e$ and $\mathbf{v}_i$ are their diffusion velocities and p_e and p_i their pressures. In accordance with statements made above, $\mathbf{B}=\mathbf{e}_yB_0$ is the external magnetic field. We shall assume that each of the components is a perfect gas. Then, for the case of equal and constant temperatures of components (which is approximately applicable for small Mach numbers and weak flows), $\nabla p_e=kT\nabla n_e$, and $\nabla p_i=kT\nabla n_i$. Noting that $n_e\approx n_i$ by virtue of quasineutrality and $v_{ey}=v_{iy}$ because of the absence of the flow towards the wall, we obtain from (3.1) and (3.2)

$$2kT\frac{\partial n_e}{\partial y}=-n_ev_{ey}\left(\frac{m_e}{\tau_e}+\frac{m_i}{\tau_i}\right),\qquad \frac{\partial}{\partial y}\,n_e(v+v_{ey})=0$$

As a result of small influence of compressibility on the flow as a whole $v=v_w=$ const and the simple diffusion equation follows

$$D_a \frac{d^2 n_e}{dy^2} - v \frac{dn_e}{dy} = 0 \qquad \left(D_a = \frac{2kT\tau_e\tau_i}{m_e\tau_i + m_i\tau_e} \approx \frac{2kT\tau_i}{m_i}\right) \tag{3.3}$$

Taking into consideration the conditions $n_e = 0$ for $y = 0$ and $n_e = n$ for $y \to \infty$ a solution of this equation exists for $v < 0$ (suction at the boundary) and has the form

$$n_e = n_\infty (1 - e^{Sy_1}) \qquad \left(S = \frac{vL}{D_a}\right)$$

in this connection the dimensionless coordinate y_1 is determined in the same manner as in sections 1 and 2. Taking into account the equation $\sigma = n_e e^2 \tau_e / m_e$, we obtain

$$\sigma = \sigma_\infty (1 - e^{Sy_1}) \tag{3.4}$$

Now we turn to Equations (1.12) and (1.13) where it is necessary to neglect terms vB^0 in comparison with $V^0 B_0$. In this case instead of (1.15) we obtain for V^0 the following equation

$$\frac{d^2V^\circ}{dy_1^2} - R\frac{dV^\circ}{dy_1} - M^{\circ 2}(1 - e^{Sy_1})(V^\circ - V_\infty{}^\circ) = 0 \tag{3.5}$$

Parameter M^{02} contains here σ_∞ instead of σ. The solution for equation (3.5) satisfying the conditions

$$V^\circ(0) = V_w, \qquad V^\circ(\infty) = V_\infty{}^\circ$$

is written in the following form using the cylindrical functions

$$V^\circ = V_\infty{}^\circ + (V_w{}^\circ - V_\infty{}^\circ)\, e^{1/_2 R y_1} \frac{J_\nu(\xi e^{1/_2 S y_1})}{J_\nu(\xi)}, \qquad \xi = -\frac{2M^\circ}{S}, \qquad \nu = -\frac{1}{S}\sqrt{R^2 + S^2\xi^2} \tag{3.6}$$

It is easy to see that expressions (3.6) when $S \to -\infty$ and (1.16) when $R_m{}^\sigma \to 0$, coincide. In the other limiting case where $|S| \ll 1$, the solution behaves monotonely near the wall, while for $y_1 \gg |S|^{-1}$ it again becomes oscillatory. Thus, an increase in the diffusion coefficient D_a leads to a shift of the first extrema u and w away from the boundary, and there is a corresponding decrease of maximum amplitudes of oscillation of the velocity profile. However, the non-monotone character of distribution of u and w is preserved for any finite D_a and $S \neq 0$.

4. Non-monotone distributions of velocities also occur in more complicated flows. In some cases when it is possible to reduce the problem to ordinary differential equations, a proof of non-monotone character is easily obtained by examining the asymptotic behavior of the solution away from the boundary. The problem of rotation of a plane in an incompressible fluid in the presence of an external homogeneous magnetic field parallel to the axis of rotation can serve as an example.

Let the plane $z = 0$ rotate with angular velocity $\Omega = \text{const}$ and let all parameters depend only on r and z. We introduce the Von Karman substitution

$$u_r = rf(z),\ u_\theta = rg(z),\ u_z = h(z),\ B_r = rF(z),\ B_\theta = rG(z),\ B_z = H(z)$$
$$p + {}^1/_8\,\pi^{-1}\mathbf{B}^2 = {}^1/_2\,p_0 r^2 + P(z),\quad p_0 = \text{const} \tag{4.1}$$

From general equations of magnetohydrodynamics we obtain the equations of motion in the form

$$\rho(f^2 + hf' - g^2) = -p_0 + \eta f'' + \frac{1}{4\pi}(F^2 + HF' - G^2) \tag{4.2}$$

$$\rho(2fg + hg') = \eta g'' + \frac{1}{4\pi}(2FG + HG') \tag{4.3}$$

$$\rho hh' = -P' + \eta h'' + \frac{1}{4\pi}HH' \tag{4.4}$$

From Ohm's law and the equation curl $\mathbf{B} = 4\pi\mathbf{j}\,/\,c$ we find

$$-rG' = \frac{\partial}{\partial r}\left(-\frac{4\pi\sigma}{c}\varphi + \frac{\alpha\mathbf{B}^2}{2}\right) + \frac{4\pi\sigma}{c^2}r(gH - hG) - \alpha r(F^2 + HF' - G^2)$$
$$rF' = \frac{4\pi\sigma}{c^2}r(hF - fH) - \alpha r(2FG + HG') \tag{4.5}$$
$$2G = \frac{\partial}{\partial z}\left(-\frac{4\pi\sigma}{c}\varphi + \frac{\alpha\mathbf{B}^2}{2}\right) + \frac{4\pi\sigma}{c^2}r^2(fG - gF) - \alpha HH'$$

Here φ is the electrical potential

$$\mathbf{B}^2 = r^2(F^2 + G^2) + H^2.$$

We assume that

$$\Phi = \frac{\alpha\mathbf{B}^2}{2} - \frac{4\pi\sigma}{c}\varphi = \Phi_0(z) + \Phi_1(z)\frac{r^2}{2}$$

Then from (4.5) it follows that

$$2G = \Phi_0' - \alpha HH' \tag{4.6}$$

$$-G' = \Phi_1 + \frac{4\pi\sigma}{c^2}(gH - hG) - \alpha(F^2 + HF' - G^2) \tag{4.7}$$

$$F' = \frac{4\pi\sigma}{c^2}(hF - fH) - \alpha(2FG + HG') \tag{4.8}$$

$$\frac{1}{2}\Phi_1' + \frac{4\pi\sigma}{c^2}(fG - gF) = 0 \tag{4.9}$$

In addition ot this, Equations div $\mathbf{v} = 0$, and div $\mathbf{B} = 0$ show that

$$f = -{}^1/_2\,h',\quad F = -{}^1/_2\,H' \tag{4.10}$$

Equations (4.4) and (4.6) are used for computation of P and Φ_0. Functions f, g, h, F, G, H and Φ_1 are determined from the system (4.2), (4.3) and (4.7) to (4.10), which after elimination of f, F and Φ_1 takes the form

$$
\begin{aligned}
\rho\left(\frac{h'^2}{4}-\frac{hh''}{2}-g^2\right) &= -p_0-\frac{\eta}{2}h'''+\frac{1}{4\pi}\left(\frac{H'^2}{4}-\frac{HH''}{2}-G^2\right)\\
\rho\,(hg'-gh') &= \eta g''+\frac{1}{4\pi}(HG'-GH')\\
-G'' &= \frac{1}{\nu_m}(g'H-hG')-\alpha\left(\frac{H'^2}{4}-\frac{HH''}{2}-G^2\right)'\\
-\frac{H''}{2} &= \frac{1}{2\nu_m}(h'H-hH')-\alpha\,(HG'-GH') \qquad (\nu_m=c^2/4\pi\sigma)
\end{aligned}
\tag{4.11}
$$

Investigation of the non-linear equations (4.11) will be carried out in the same manner as in section 2, i.e. by finding exponential solutions of the corresponding linearized equations. Assuming that $h = h_\infty + A_h e^{\gamma z}$, $H = H_\infty + A_H e^{\gamma z}$,... etc., we obtain for γ the equation

$$
\begin{aligned}
&\gamma^2\left[\left(\gamma-\frac{h_\infty}{\nu_m}+2\alpha G_\infty\right)\left(\gamma-\frac{h_\infty}{\nu}\right)-\frac{H_\infty^{\ 2}}{4\pi\eta\nu_m}\right]^2+\gamma^4\alpha^2H_\infty^{\ 2}\left(\gamma-\frac{h_\infty}{\nu}\right)^2+\\
&+4\left[\left(\gamma-\frac{h_\infty}{\nu_m}+2\alpha G_\infty\right)\frac{g_\infty}{\nu}+\frac{H_\infty G_\infty}{4\pi\eta\nu_m}\right]^2+\frac{4g_\infty\gamma^2}{\nu^2}\alpha H_\infty^{\ 2}\left(\alpha g_\infty+\frac{H_\infty^{\ 2}}{4\pi\rho\nu_m}\right)+\\
&+\gamma^2\left(\gamma-\frac{h_\infty}{\nu}\right)\frac{\alpha G_\infty H_\infty^{\ 2}}{\pi\eta\nu_m}=0
\end{aligned}
\tag{4.12}
$$

For $\alpha = 0$, and $G_\infty = g_\infty = 0$ we obtain from here two, different from zero, real values for γ, which correspond to monotone variations of velocity [7].

If $\alpha \neq 0$, while $G_\infty = g_\infty = 0$, then the equation for γ assumes the form

$$
\left[\left(\gamma-\frac{h_\infty}{\nu_m}\right)\left(\gamma-\frac{h}{\nu}\right)-\frac{H_\infty^{\ 2}}{4\pi\eta\nu_m}\right]^2+\gamma^2\alpha^2H_\infty^{\ 2}\left(\gamma-\frac{h_\infty}{\nu}\right)^2=0
$$

and gives four complex values of γ, which are identical to $\gamma_{1,2}$, and $\tilde{\gamma}_{1,2}$, found in section 1 for the problem of translational flow of an incompressible fluid above a permeable plane, within the accuracy of the method. Thus, if the azimuth components of velocity and of the field are absent at infinity, then the geometry of the problem has no effect on asymptotic properties of the solution. When one of the equalities $G_\infty = 0$, or $g_\infty = 0$, is not fulfilled, for example in the case $H_\infty = 0$, $g_\infty \neq 0$, from (4.12) follows

$$
\gamma_{1,2} = h_\infty / \ n - 2\alpha G_\infty, \qquad \gamma_{3-6} = h_\infty / 2\nu \pm \sqrt{(h_\infty/2\nu)^2 \pm 2ig_\infty/\nu} \tag{4.13}
$$

The last four roots do not depend on α, ν_m, and G_∞ and are complex for all $g_\infty \neq 0$

When $\alpha = 0$, and $\rho g_\infty h_\infty = - H_\infty G_\infty / 4\pi$ non-zero roots of (4.12)

$$
\gamma_{1-4} = \frac{h_\infty}{2}\left(\frac{1}{\nu}+\frac{1}{\nu_m}\right) \pm \left[\frac{h_\infty^{\ 2}}{4}\left(\frac{1}{\nu}+\frac{1}{\nu_m}\right)^2 \pm \frac{2ig_\infty}{\nu} - \frac{\rho h_\infty^{\ 2} - H_\infty^{\ 2}/4\pi}{\eta\nu_m}\right]^{1/2} \tag{4.14}
$$

are also complex for $g_\infty \neq 0$.

Thus, in contrast with the problems in section 1 to 3, the periodic structure of the flow above a rotating disc may not only be caused by the anisotropy of conductivity, but may also occur when a non-conducting fluid rotates at infinity. This conclusion is justified under the assumption that the system (4.1) has a solution with $g_\infty \neq 0$. The existence of such solutions requires a separate proof. Periodic distributions of velocities in the case of rotation of a non-conducting and a weakly conducting ($\alpha = 0$, $R_m \ll 1$) fluid above a stationary plane were discovered in [8] and [9], respectively, where the solutions for boundary layer equations were also constructed.

BIBLIOGRAPHY

1. Fay, J.A., Plasma boundary layers. 'Magnetohydrodynamics', Evanston, Northwestern Univ. Press, p. 337, 1962.

2. Sakhnovskii, E.G., Uchet anizotropii provodimosti v magnitogidrodinamicheskoi zadache Releia (Consideration of anisotropy of conductivity in the magnetohydrodynamic problem of Rayleigh). Zh. tekhn. fiz. Vol. 33, No. 5, pp. 631-635, 1963.

3. Kulikovskii, A.G. and Liubimov, G.A., Magnitnaia gidrodinamika (Magnetohydrodynamics). M., Fizmatgiz, 1962.

4. Semenova, I.P., Statsionarnoe techenie anizotropno-provodiashchei zhidkosti mezhdu parallel'nymi poristymi stenkami (Steady flow of an anisotropically-conducting fluid between parallel porous walls). PMTF No. 4, pp. 118-121, 1964.

5. Regirer, S.A., Techenie viazkoi provodiashchei zhidkosti v oblastiakh s pronitsaemymi granitsami v prisutstvii magnitnogo polia (Flow of a viscous conducting fluid in regions with permeable boundaries in the presence of a magnetic field). Sb. 'Voprosy magnitnoi gidrodinamiki' (Compendium 'Problems in magnetohydrodynamics') Vol. 2, Riga Izd-vo AN LatvSSR, pp. 107-112, 1962.

6. Liubimov, G.A., Struktura magnitogidrodinamicheskoi udarnoi volny v gaze s anizotropnoi provodimost'iu (Structure of a magnetohydrodynamic shock wave in a gas with anisotropic conductivity). *PMM* Vol. 25, No. 2, pp. 179-186, 1961.

7. Sychev, V.V., O dvizhenii viazkoi elektroprovodnoi zhidkosti pod deistviem vrashchaiushchegosia diska v prisutsvii magnitnogo polia (On the motion of a viscous electrically conducting fluid, under the influence of a rotating disk and in the presence of a magnetic field). *PMM* Vol. 24, No. 5, pp. 906-915, 1960.

8. Boedewadt, U.T., Die Drehstroemung ueber festem Grunde. Z. Angew. Math. und Phys., Vol. 20, No. 3, 1940.

9. King, W.S. and Lewellen, W.S., Boundary-layer similarity solutions for rotating flows with and without magnetic interaction. Phys. Fluids Vol. 7, No. 10, 1964.

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